

Substitute $y = -2x$ in $x^2 + xy + y^2 = 3$.

$$x^2 + x(-2x) + (-2x)^2 = 3$$

$$x^2 - 2x^2 + 4x^2 = 3$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = 1 \text{ or } -1$$

From [1], when $x = 1$, $y = -2$

when $x = -1$, $y = 2$

The curve has stationary points at $(1, -2)$ and $(-1, 2)$.

[The second derivative will not be used here as the second derivative for implicit functions is not on the syllabus for C4.]

EXERCISE 1

- 1 For each curve find $\frac{dy}{dx}$ in terms of x and y .

a $x^2 + y^2 = 9$

b $xy = 4$

c $x^2 + xy + y^2 = 0$

d $x^2 + y^2 - 6x + 8y = 0$

e $x^2 + 3y^2 - 4y = 0$

f $y^3 = x^2 + 10$

g $3x^3 + 4xy^2 + y^3 = 0$

- 2 Find $\frac{dy}{dx}$ for the following curves at the points indicated:

a $x^2 + y^2 = 10$ $(3, 1)$

b $xy = 9$ $(3, 3)$

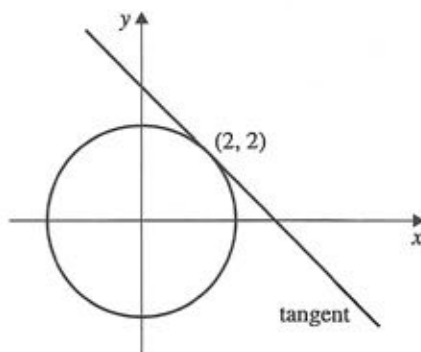
c $x^2 + y^2 + 4x = 9$ $(1, 2)$

d $y^2 = x^3$ $(4, 8)$

- 3 The diagram shows a circle with equation $x^2 + y^2 = 8$.

By finding $\frac{dy}{dx}$, calculate the gradient of the tangent to the circle at the point $(2, 2)$.

Using the equation $y - y_1 = m(x - x_1)$, find the equation of the tangent at the point $(2, 2)$.



- 4 Find the equation of the normal to $x^2 + y^2 = 8$ at the point $(2, 2)$.
[Hint: Remember that gradient of tangent $\times$ gradient of normal $= -1$]
- 5 Find the equation of the tangent to the curve $xy = 4$ at the point $(2, 2)$.
- 6 Find the equation of the normal to the curve $xy = 4$ at the point $(-2, -2)$.
- 7 Find the equation of the tangent to the circle $x^2 + y^2 - 2x + 4y = 20$ at the point $(4, 2)$.

8 Find the equation of the normal to the curve $x^2 + xy + y^2 = 3$ at the point $(1, 1)$.

9 For the curve $\ln y + x^2 - x = 0$, show that $\frac{dy}{dx} = y(1 - 2x)$.

10 Find the gradient of each curve at the point given.

a $x^3 + xy + y^3 = 20$ $(2, 2)$

b $2x^3 - 3x^2y - y^2 = 0$ $(-1, -2)$

c $\ln y + x^2 = 4$ $(2, 1)$

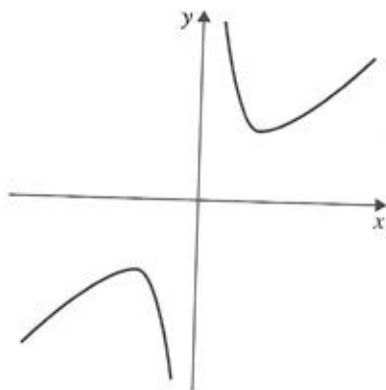
d $3 \ln y + 2x^3 = 16$ $(2, 1)$

11 The diagram shows the graph of $xy - x^2 = 4$.

a Show that when $\frac{dy}{dx} = 0$, $y = 2x$.

b Substitute $y = 2x$ into the equation $xy - x^2 = 4$ to find the x coordinates of the two points where the gradient is zero.

c Find the y coordinates of the two turning points.



12 Show that the stationary points on the curve $x^2 + 4xy - 2y^2 + 24 = 0$ occur when $x = -2y$. Hence find the stationary points.

13 A curve is defined by the equation $x^2 - 4x - 2xy + 3y^2 - 2 = 0$.

a Find $\frac{dy}{dx}$ in terms of x and y .

b Show that the equation on which the stationary points must lie is $y = x - 2$.

c Show that one of the stationary points is $(1, -1)$ and find the co-ordinates of the other one.

14 a Show that the tangents to the curve $y^2 - 16x - 2y = 47$, at the points $(-2, -3)$ and $(13, 17)$, are perpendicular.

b Find the co-ordinates of their points of intersection.

15 Calculate $\frac{dy}{dx}$ for the following curves at the given points:

a $x^2 - 7xy + y^2 = -29$ $(2, 3)$

b $xy^2 - 7 \sin y + x = 5$ $(5, 0)$

c $y + x \ln y + x^2 = 5$ $(2, 1)$

d $e^x y + y^2 = 6$ $(0, 2)$

e $(x - y)^2 = 4xy - 7$ $(2, 1)$

16* a Given that $x = \cos y$,

i find $\frac{dy}{dx}$ in terms of y (by differentiating both sides with respect to x)

17* a Given that $x = \tan y$,

i find $\frac{dy}{dx}$ in terms of y (by differentiating both sides with respect to x)

ii find $\frac{dy}{dx}$ in terms of x (using the fact that $1 + \tan^2 y = \sec^2 y$)

b Use part a to find $\frac{dy}{dx}$ in terms of x when $y = \tan^{-1} x$.

18* Given $y = x^x$, find $\frac{dy}{dx}$ in terms of x .

19* Find $\frac{dy}{dx}$ in terms of x and y .

a $e^y = x$

b $e^{xy} = 2$

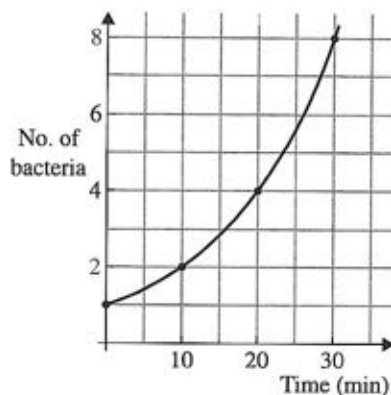
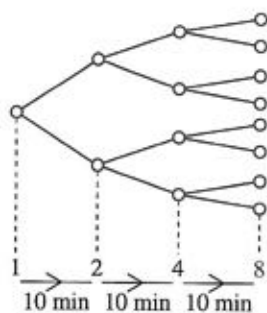
c $\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$

d $\cos x \sin y = \frac{1}{2}$

e $\sin y + \cos y = x$

9.2 Exponential growth and decay

Under certain conditions, the number of bacteria in a particular culture doubles every 10 minutes.



This is an example of *exponential growth*. The number of bacteria is multiplied by the same number in equal periods of time.

After t minutes the number of bacteria N is given by $N = 2^{\frac{t}{10}}$. You can check this formula by substituting $t = 0, 10, 20$ and so on.

Radioactive materials like uranium or plutonium decay very, very slowly. The *half life* is the length of time for half the material to decay. Suppose uranium has a half life of 40 years and there is 100 g of it now. In 40 years there would be 50 g. After another 40 years there would still be 25 g and so on.

After t years the mass of uranium m is given by $m = 100 e^{-0.0173t}$. [Check by substituting $t = 40, 80$.] This is an example of exponential decay.

Example 5

Solve the inequality $0.3^x < 0.02$.

Take logarithms (base 10).

$$\begin{aligned}\log 0.3^x &< \log 0.02 \\ x \log 0.3 &< \log 0.02\end{aligned}$$

Be careful here. Since $\log 0.3$ is a negative number, the inequality must be reversed when dividing both sides by $\log 0.3$.

$$\begin{aligned}x &> \frac{\log 0.02}{\log 0.3} \\ x &> 3.25, \text{ correct to three significant figures.}\end{aligned}$$

EXERCISE 2

1 Find x , correct to 3 significant figures.

a $4^x = 11$

b $2^x = 19$

c $5^{x-1} = 100$

d $e^x = 19$

e $5e^{2x} = 32$

f $3e^{-x} = 11$

g $3(7^{\frac{x}{2}}) = 20$

h $9^{3x-1} = 1$

i $\ln x = 3$

j $\ln 2x = 5$

k $2 \ln x = -1$

l $\ln x^2 - \ln 2x = 4$

2 Given $m = 40e^{-4t}$, find the value of m when $t = 0.1$.

3 Given $P = 6000e^{\frac{t}{100}}$, find the value of P when $t = 20$.

4 Given $x = 200 \times 4^{2t-1}$, find the value of x when $t = 1.2$.

5 Find $\frac{dy}{dx}$ in the following.

a $y = 3^x$

b $y = 5^{x-1}$

c $y = 3(4^x)$

d $y = 2^{x^2}$

e $y = e^{-\frac{x^2}{3}}$

6 a Find $\frac{d}{dt}(7^t)$

b Find $\frac{d}{dt}(4^t)$

c Find $\frac{d}{dt}\left(\frac{1}{3^t}\right)$

7 Find $\frac{d\theta}{dt}$ in the following.

a $\theta = 150e^{-\frac{t}{10}}$

b $\theta = 100 - e^{-2t}$

c $\theta = 10 \times 6^t$

8 A substance is decaying exponentially. Its mass, m grams, after t years is given by $m = 1000e^{-0.02t}$.

a Write down the initial value of m .

b Find the value of t when $m = 400$.

9 Find t , correct to 3 significant figures.

a $5e^{-0.2t} = 11$

b $1000 = 2500 - 1800e^{-0.1t}$

c $450 = 600 - 500e^{-0.01t}$

- 10 At time t hours the temperature $\theta^\circ\text{C}$ of a cooling body is given by $\theta = 80 e^{kt}$, where k is a constant.
- Given that $\theta = 40^\circ\text{C}$ when $t = 10$ hours, find the value of k .
 - Find the value of θ when $t = 40$.
- 11 At time t minutes the temperature $\theta^\circ\text{C}$ of a cooling furnace is given by $\theta = 2500 e^{-0.9t}$.
Find the rate of decrease of the temperature when $t = 2$.
- 12 After t years the mass m grams of a piece of radioactive material is given by $m = 1000 \times 2^{-0.002t}$.
- State the value of m when $t = 0$.
 - Find the value of t when the material has decayed to one tenth of its original mass.
- 13 A population of flies P is given by the formula $P = A e^{-kt}$, where t is the time in days measured from a time when $P = 2000$.
- Write down the value of A .
 - Given that $P = 500$ when $t = 5$, show that $k = \frac{1}{5} \ln 4$.
 - Find the value of P when $t = 8$ days.
- 14 A radioactive substance decays so that after t days the amount remaining is 0.75^t units.
- How many units remain after 10 days?
 - After how many days will the amount of the substance remaining be less than 0.1 units?
- 15 A radioactive substance is decaying exponentially. Its mass, m grams, after t years is given by $m = 300 e^{-0.005t}$.
- Find the value of t when $m = 100$.
 - Find the rate at which the mass is decreasing when $t = 20$.
- 16 Make t the subject in the following formulae.
- $m = a e^{3t}$
 - $v = \frac{b}{10} e^{-t}$
 - $P = 150 e^{t^2}$
- 17 At time t minutes after being switched on, the temperature of a furnace $\theta^\circ\text{C}$ is given by $\theta = 2000 - 1800 e^{-0.1t}$.
- State the value which θ approaches after a long time.
 - Find the time taken to reach a temperature of 1500°C .
 - Find the rate at which the temperature is increasing when $t = 10$.
- 18 At time t minutes after being switched on, the temperature of an oven $\theta^\circ\text{C}$ is given by $\theta = 190 - 175 e^{-0.08t}$.
- State the value which θ approaches after a long time.
 - Find the time taken to reach a temperature of 160°C .
 - Find the rate at which the temperature is increasing when $\theta = 50^\circ\text{C}$.

Example 2

The volume of a sphere is increasing at a rate of $100 \text{ cm}^3 \text{ s}^{-1}$. Find the rate of increase of the radius of the sphere at the instant when the radius is 3 cm.

For a sphere, $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = 4\pi r^2, \text{ and when } r = 3, \frac{dV}{dr} = 36\pi.$$

We are given $\frac{dV}{dt} = 100$ and we want to find $\frac{dr}{dt}$.

By the chain rule, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$$100 = 36\pi \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{100}{36\pi} = \frac{25}{9\pi} \text{ cm s}^{-1}.$$

Example 3

A hollow right circular cone, whose height of the cone is twice the radius, is held with its vertex downwards and water is poured in at a rate of $5\pi \text{ cm}^3 \text{ s}^{-1}$. Find the rate of rise of the water level when the depth is 2 cm.

The volume of the water in the cone V when the height is h is given by $V = \frac{1}{3}\pi r^2 h$. We are given that $h = 2r$.

$$\therefore V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

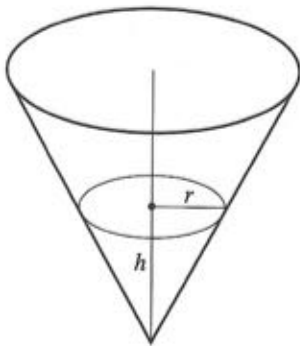
$$\frac{dV}{dh} = \frac{1}{4}\pi h^2, \text{ and when } h = 2, \frac{dV}{dh} = \frac{1}{4} \times \pi \times 2^2 = \pi$$

We are given $\frac{dV}{dt} = 5\pi \text{ cm}^3 \text{ s}^{-1}$ and we want to find $\frac{dh}{dt}$.

By the chain rule, $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$$5\pi = \pi \times \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = 5 \text{ cm s}^{-1}$$

The water level is rising at 5 cm s^{-1} .



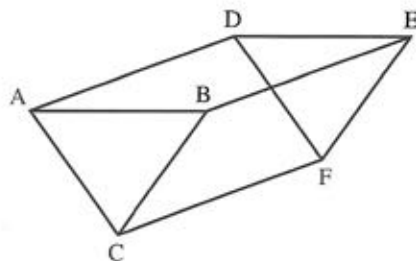
EXERCISE 3

Give answers to three significant figures where necessary.

- 1 Given that $y = x^2$ and that x is increasing at a rate 0.1 cm s^{-1} , find the rate at which y is increasing when $x = 3 \text{ cm}$.
- 2 If the radius of a circle increases at a rate of 0.5 cm s^{-1} , find the rate at which the area of the circle increases when the radius is 10 cm .

- 3 The area of a square is increasing at a rate of $4 \text{ cm}^2 \text{ s}^{-1}$. Find the rate of increase of each side when each side is of length 10 cm.
- 4 A cube is expanding. Each edge of the cube increases at a rate of 0.01 cm s^{-1} . When the edges of the cube are 8 cm find the rate of increase of
 - a the volume
 - b the surface area of the cube.
- 5 If the radius of a sphere is increasing at 3 cm s^{-1} , find the rate at which the volume is increasing when the radius is 5 cm.
- 6 The area of a circular ink blot is increasing at a rate of $0.8 \text{ cm}^2 \text{ s}^{-1}$. Find the rate at which the radius is increasing when the radius of the blot is 2 cm.
- 7 The radius of a sphere is increasing at the rate of 2 cm s^{-1} . Find, in terms of π , the rate of increase of the volume when the radius is 10 cm.
- 8 The side length of a cube is increasing at the rate of 3 cm s^{-1} . Find the rate of increase of the volume when the length of a side is 5 cm.
- 9 The volume of a cube is increasing at the rate of $96 \text{ cm}^3 \text{ s}^{-1}$. Find the rate of change of the side of the base when its length is 4 cm.
- 10 The surface area of a sphere is increasing at the rate of $40\pi \text{ cm}^2 \text{ s}^{-1}$. Find the rate of change of the radius when its length is 1 cm. [The surface area of a sphere, $S = 4\pi r^2$]
- 11 A hollow right circular cone, whose height of the cone is twice the radius, is held with its vertex downwards and water is poured in at a rate of $4 \text{ cm}^3 \text{ s}^{-1}$.
 - a Show that when the depth of the water is h the volume in the cone is $V = \frac{\pi h^3}{12}$.
 - b Find, in terms of π , the rate of rise of the water level when the depth is 2 cm.

- 12 The empty water container shown is a right prism with $AD = 3 \text{ m}$ and $AC = AB = BC = 1 \text{ m}$. It lies with CF on horizontal ground and with $ABDE$ horizontal.



- a Show that the volume (in m^3) in the container when the water has height h is $V = \sqrt{3}h^2$.

Water is poured into the container in such a way that the height is rising at $5\sqrt{3} \text{ cm s}^{-1}$ when the height of the water is 50 cm.

- b Find the rate (in $\text{m}^3 \text{ s}^{-1}$) at which the volume of water is increasing at this time.

- 13 Given that $y^2 = x^3$ and that x is increasing at a rate of 0.5 units per second, find the rate at which y is increasing when $x = 4$ units.
- 14 A liquid drips onto a surface and forms a circular film of uniform depth 0.1 cm. The liquid drips at a rate of $0.2 \text{ cm}^3 \text{ s}^{-1}$. When the radius of the film is 3 cm find the rate at which the radius is increasing.
- 15* a Show that the volume obtained when the part of the curve $y = \sqrt{9 - x^2}$ that is bounded by the x -axis and the line $x = 3 - h$ is rotated about the x -axis is $\frac{\pi h^2}{3}(9 - h)$.
- Water is poured at a rate of $18 \text{ cm}^3 \text{ s}^{-1}$ into a hemisphere whose radius is 3 cm.
- b Find, in terms of π , the rate of rise of the water level when the depth is just about to fill up.

REVIEW EXERCISE 9

- Find $\frac{dy}{dx}$ (using implicit differentiation), in terms of x and y .
 - $y^2 + y = x$
 - $y + xy + y^2 = 2$
 - $x^2 + y^2 = 2$
- For the following curves express $\frac{dy}{dx}$ in terms of x and y .
 - $x^2 + 3y^2 = 7$
 - $4x^2 - y^3 + 2x + 3y = 0$
 - $3x^2 + 4xy - 5y^2 = 20$
 - $xy^3 + x^3y = x - y$
 - $\sin x + 2 \cos y = 1$
 - $x^2y^2 - y = 2$
- Find the gradient of the curve $x^3 - 2y^3 = 3xy$ at the point $(2, 1)$.
- A curve is defined by the equation $x^2 + 7xy + 3y^2 = 27$.
 - Find $\frac{dy}{dx}$ in terms of x and y .
 - Hence find the value of $\frac{dy}{dx}$ at the point $(1, 2)$ on the curve.
- Find, in the form $ax + by + c = 0$, the equation of the tangent to the curve $x^2 + xy + 4y^2 = 16$ at the point $(3, 1)$.
- Find, in the form $ax + by + c = 0$, the equation of the normal to the curve $x^2y^2 = x^2 + 5y^2$ at the point $(3, \frac{3}{2})$.
- For the curve $x^2 + 4y^2 - 4x - 8y + 4 = 0$, express $\frac{dy}{dx}$ in terms of x and y .
 - What can you say about the gradient of the curve at the points $(2, 2)$ and $(0, 1)$ on the curve?
- Show that $(-1, 3)$ and $(0, 0)$ are stationary points on the curve $3x^2 + 2xy - 5y^2 + 16y = 0$.

- 9 For the curve $\ln y = 3x^2 + 4$, show that $\frac{dy}{dx} = 6xy$.
- 10 a For the curve $3 \sin x + 2 \cos y = \sqrt{3}$ show that $\frac{dy}{dx} = \frac{3 \cos x}{2 \sin y}$.
 b Find the gradient of the curve at the point $\left(0, \frac{\pi}{6}\right)$.
- 11 Find the gradient of each curve at the point given.
 a $y + x \ln y + x^2 = 10$ at $(3, 1)$
 b $e^x y + 2y = 12$ at $(0, 4)$
- 12 Find $\frac{dy}{dx}$ in terms of x .
 a $y = 8^x$ b $y = 2 \times 3^x$ c $y = 4^{x^2}$
 d $y = 4e^{-3x}$ e $y = x^x$
- 13 Show that the gradient of the curve $y = 3^x$ at the point $(1, 3)$ is $\ln 27$.
- 14 A radioactive substance is decaying exponentially. Its mass, m grams, after t years is given by $m = 300e^{-0.05t}$.
 a Write down the initial value of m .
 b Find the value of t when $m = 100$.
- 15 At time t minutes the temperature $\theta^\circ\text{C}$ of a cooling oven is given by $\theta = 450e^{-0.8t}$.
 Find the rate of decrease of the temperature when $t = 10$.
- 16 At time t hours the temperature $\theta^\circ\text{C}$ of a cooling body is given by $\theta = 100e^{kt}$, where k is a constant.
 a Given that $\theta = 60$ when $t = 2$, find the value of k .
 b Find the value of θ when $t = 6$.
- 17 Make t the subject in the following formulae.
 a $m = Ae^{2t}$ b $s = \frac{a}{4}e^{-t}$ c $v = 20e^{\theta}$
- 18 A substance is decaying exponentially. After t years, its mass m grams is given by $m = 1000e^{-0.01t}$.
 a Find the value of t when $m = 800$.
 b Find the rate at which the mass is decreasing when $t = 10$.
- 19 For a body falling vertically, its velocity $v \text{ m s}^{-1}$ after t seconds is given by $v = 50(1 - e^{-0.2t})$.
 i State the value which the velocity approaches after a long time.
 ii Find the time for the velocity to reach 30 m s^{-1} .
 iii Find the rate at which the velocity is increasing when the velocity reaches 30 m s^{-1} .

- 20 The population P of a particular species is modelled by the formula

$$P = Ae^{kt},$$

where t is the time in years measured from a date when $P = 1000$.

- Write down the value of A .
 - Given that $P = 2000$ when $t = 5$, show that $k \approx 0.139$.
 - Find the value of the population 10 years after the initial date.
- 21 The radius of a sphere is increasing at the rate of 1 cm s^{-1} . Find, in terms of π , the rate of increase of the volume when the radius is 5 cm.
- 22 The side length of a cube is increasing at the rate of 2 cm s^{-1} . Find the rate of increase of the volume when the length of a side is 6 cm.
- 23 The surface area S of a sphere of radius r is given by the formula $S = 4\pi r^2$. Find the rate of increase of the surface area when $r = 4 \text{ cm}$, given that the radius increases at 2 cm s^{-1} .
- 24 The volume of a cube is increasing at the rate of $81 \text{ cm}^3 \text{ s}^{-1}$. Find the rate of change of the side of the base when its length is 3 cm.

- 25 The surface area of a sphere is increasing at the rate of $2\pi \text{ cm}^2 \text{ s}^{-1}$. Find the rate of change of the radius when its length is 1 cm.

- 26 A hollow right circular cone, whose height is three times the radius, is held with its vertex downwards and water is poured in at a rate of $10 \text{ cm}^3 \text{ s}^{-1}$.

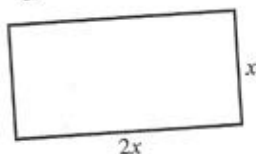
- Show that when the depth of the water is h the volume in the cone is

$$V = \frac{\pi h^3}{27}.$$

- Find, in terms of π , the rate of rise of the water level when the depth is 2 cm.
- 27 When a metal cube is heated, its volume, $V \text{ cm}^3$, increases at a constant rate of $3.7 \times 10^{-6} \text{ cm}^3 \text{ s}^{-1}$. Each edge of the cube has length $x \text{ cm}$ at time t seconds.
- Find $\frac{dx}{dt}$ when $x = 2$.
 - Find the rate of increase of the total surface area of the cube in $\text{cm}^2 \text{ s}^{-1}$, when $x = 2$.

- 28 The area of a circle is increasing at a rate of $6 \text{ cm}^2 \text{ s}^{-1}$. Find the rate of increase of the circumference when the radius is 3 cm.

- 29 The area and perimeter of the rectangle shown are denoted by A and P respectively. Find $\frac{dP}{dt}$, given $\frac{dA}{dt} = 16 \text{ cm}^2 \text{ s}^{-1}$ at the instant when $x = 2 \text{ cm}$.



EXAMINATION EXERCISE 9

- 1 Find the gradient of the curve $x^3 + y^3 = 19$ at the point where $x = -2$. [OCR]
- 2 i For the curve $2x^2 + xy + y^2 = 14$, find $\frac{dy}{dx}$ in terms of x and y .
 ii Deduce that there are two points on the curve $2x^2 + xy + y^2 = 14$ at which the tangents are parallel to the x -axis, and find their coordinates. [OCR]
- 3 A curve has equation $\frac{x^2}{9} + \frac{y^2}{25} = 1$.
 a Find the y -coordinates of the two points on the curve at which the x -coordinate is 2.
 b Find the values of the gradient of the curve at these two points, giving your answers to two significant figures. [AQA]
- 4 A curve has equation $7x^2 + 48xy - 7y^2 + 75 = 0$.
 A and B are two distinct points on the curve. At each of these points the gradient of the curve is equal to $\frac{2}{11}$.
 a Use implicit differentiation to show that $x + 2y = 0$ at the points A and B.
 b Find the coordinates of the points A and B. [EDEXCEL]
- 5 The curve C has equation $5x^2 + 2xy - 3y^2 + 3 = 0$. The point P on the curve C has coordinates (1, 2).
 a Find the gradient of the curve at P.
 b Find the equation of the normal to the curve C at P, in the form $y = ax + b$, where a and b are constants. [EDEXCEL]
- 6 A curve is given by the equation

$$3(x + 1)^2 - 9(y - 1)^2 = 32.$$
 a Find the coordinates of the two points on the curve at which $x = 3$.
 b Find the gradient of the curve at each of these two points. [AQA]
- 7 a Given that $a^x \equiv e^{kx}$, where a and k are constants, $a > 0$ and $x \in \mathbb{R}$, prove that $k = \ln a$.
 b Hence, using the derivative of e^{kx} , prove that when $y = 2^x$,

$$\frac{dy}{dx} = 2^x \ln 2$$
 c Hence deduce that the gradient of the curve with equation $y = 2^x$ at the point (2, 4) is $\ln 16$. [EDEXCEL]
- 8 a A quantity N is growing exponentially. The table shows values of N at different times t . Determine the values of P and q .

t	20	30	40	q
N	50	100	P	800

 b A radioactive substance is decaying exponentially. Its mass, m grams, after t years is given by

$$m = 600 e^{-0.0035t}.$$
 i Find the value of t for which $m = 300$.
 ii Find, correct to 2 significant figures, the rate at which the mass is decreasing when $t = 80$. [OCR]

- 9 A radioactive substance is decaying exponentially. Initially its mass is 480 g. Its mass, M grams, at a time t years after the initial observation is given by

$$M = 480 e^{kt},$$

where k is a constant. When $t = 330$, the mass of the substance will be 240 g.

- State the value of the mass when $t = 660$.
 - Determine the value of k .
 - Find the rate at which the mass will be decreasing when $t = 200$. [OCR]
- 10 The decay of a radioactive substance can be modelled by the equation

$$m = m_0 e^{-kt},$$

where m grams is the mass at time t years, m_0 grams is the initial mass, and k is a constant.

- The time taken for a sample of the radioactive substance strontium 90 to decay to half of its initial mass is 28 years. Show that the value of k is approximately 0.024 755.
 - A sample of strontium 90 has a mass of 1 gram. Assuming this mass has resulted from radioactive decay, use the model to find the mass this sample would have had 100 years ago. Give your answer to three significant figures. [AQA]
- 11 The mean annual temperature of the water in a certain large lake is expected to increase due to climatic change. A model giving the mean annual temperature, $\theta^\circ\text{C}$, at a time t years after the first observation is

$$\theta = 0.0006t^2 + 0.05t + 4.3.$$

The number of crustaceans in the lake depends on the water temperature. A model giving the number, N , of crustaceans is

$$N = 320 e^{0.4\theta}.$$

- According to the models, how many crustaceans were present in the lake when the first observation was made?
 - By first writing down expressions for $\frac{d\theta}{dt}$ and $\frac{dN}{d\theta}$, find the rate at which the models predict that the number of crustaceans will be increasing when $t = 30$. [OCR]
- 12 Observations were made of the number of bacteria in a certain specimen. The number N present after t minutes is modelled by the formula

$$N = Ac^t,$$

where A and c are constants.

Initially there are 1000 bacteria in the specimen.

- Write down the value of A .
- Given that there are 12 000 bacteria after 60 minutes, show that the value of c is 1.0423 to four decimal places.
- Express t in terms of N .
 - Calculate, to the nearest minute, the time taken for the number of bacteria to increase from one thousand to one million. [AQA]

- 13 The volume of a sphere is increasing at a rate of $75 \text{ cm}^3 \text{ s}^{-1}$. Find the rate at which the radius is increasing at the instant when the radius of the sphere is 15 cm. Give your answer correct to 2 significant figures.

[The volume, V , of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.] [OCR]

- 14 At a time t minutes, a circular puddle has radius r cm and area $A \text{ cm}^2$.

a Find $\frac{dA}{dr}$ in terms of r .

b The radius is increasing at a rate of 3 cm per minute.

Find the rate at which the area is increasing at the instant when the radius is 50 cm. Give your answer to the nearest cm^2 per minute. [AQA]

- 15 a Given that $y = 5e^{3x} + \ln(2 - 5x)$, find an expression for $\frac{dy}{dx}$.

b A water tank is such that, when the depth of water is h cm, the volume, $V \text{ cm}^3$, of water in the tank is given by $V = 28h^3$. Water is flowing into the tank at a constant rate of 400 cm^3 per minute. Find the rate at which the depth of water is increasing at the instant when $h = 10$. [OCR]

- 16 As part of a laser light show, a rectangular shape continuously changes its dimensions whilst its perimeter remains constant at 20 metres.

a At time t seconds, the length of one side of the rectangle is x metres.

i Show that the area $A \text{ m}^2$ of the rectangle is given by

$$A = x(10 - x).$$

ii Find $\frac{dA}{dx}$.

b Find the rate at which the area of the rectangle is increasing when the length of one of the sides is 4 metres and is increasing at a rate of 0.5 m s^{-1} . [AQA]

- 17 A scientific study involves monitoring the growth of a particular species of weed. The area of ground affected by the weed after t years is $A \text{ m}^2$. Two models giving A in terms of t are as follows.

Model 1: $A = 10 + 8 \ln(t + 1)$.

Model 2: $A = 5(4 + 3t)^{\frac{1}{2}}$.

i For Model 1:

a find the value of A when $t = 15$, giving your answer correct to 3 significant figures;

b find by differentiation the rate at which the area affected by the weed is increasing when $t = 15$.

ii For Model 2:

a find the value of A when $t = 15$;

b find by differentiation the rate at which the area affected by the weed is increasing when $t = 15$.